Towards the use of Simplification Rules in Intuitionistic Tableaux

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Abstract. By replacement it is meant the substitution of one or more occurrences of a formula with an equivalent one. In automated deduction this can be useful to reduce the search space. In tableau calculi for classical and modal logics this technique is known as *simplification* and consists in replacing a formula with a logical constant ($\top$ or $\bot$). Recently, this idea has been applied to Intuitionistic Logic. This work in progress investigates further conditions on the applicability of Simplification in Intuitionistic Logic.

1 Introduction

It is well-known that the problem of deciding propositional Intuitionistic Logic is PSPACE-complete [7]. As a consequence, to perform automated theorem proving strategies reducing the search space are needed. A technique is *replacement*, that is the substitution of one or more occurrences of a formula with an equivalent one [3].

In the framework of tableau calculi for classical and modal logics, it has been described a technique, known as *simplification* [4], consisting in replacing every occurrence of a formula proved to be true with the logical constant $\top$ and replacing a formula proved to be false with $\bot$. As an example, if A can be replaced with $\top$, we can simplify the formula $A \vee B$ in $\top \vee B$, which turns out to be equivalent to $\top$. In the tableau systems for classical logic the notions of provable and unprovable are codified by means of the *signs* $\mathbf{T}$ and $\mathbf{F}$ [6]. It is well-known that the sign (polarity) of a formula determines also the sign of every occurrence of its subformulas. Moreover, if the sign of a propositional variable occurring in a set of signed formulas is always $\mathbf{T}$ (respectively $\mathbf{F}$), then such a variable is equivalent to $\top$ (respectively $\bot$).

Also in the intuitionistic setting the sign of a formula determines the sign of every subformula. Differently from classical logic, the signs $\mathbf{T}$ and $\mathbf{F}$ are not dual, in particular $\mathbf{F}A$ does not imply that A is equivalent to $\bot$. Thus, simplifications can be performed only if further conditions are satisfied. In this paper we provide some simplification rules. In Sections 4 and 5 we introduce the rules **T**-permanence, **T**$\neg$-permanence and **F**-permanence that allows to replace, under suitable conditions, propositional variables with $\top$ and $\bot$. After the substitutions, we can apply the known boolean simplification rules and reduce the size of the set of formulas to be decided. The simplification rules we identify in this paper derive from a semantical analysis of validity of formulas in Kripke models.

We remark that our simplification rules are essentially independent from the tableau calculus at hand. Moreover, these rules are invertible. This means that we can apply them at any point of a proof search strategy without affecting its completeness. Finally, via the usual translation, these rules can also be applied in implementations based on sequent calculi.

This is a work in progress. We have implemented a Prolog prototype and we have compared its performances with PITP [1], which is, by now, the fastest prover for Intuitionistic Propositional Logic on the formulas of the benchmark ILTP Library [5]. As discussed in Section 6, the results are encouraging. We plan to continue the investigation by studying further criteria for variable replacement.

2 Notation and Preliminaries

We consider the propositional language $\mathcal{L}$ based on a denumerable set of propositional variables $\mathcal{PV}$, the logical connectives $\neg$, $\wedge$, $\vee$, $\rightarrow$, the logical constants $\top$ and $\bot$. We recall the main definitions about Kripke semantics (see e.g. [2] for more details). A Kripke model for $\mathcal{L}$ is a structure $\underline{K} = \langle P, \leq, \rho, \Vdash \rangle$, where $\langle P, \leq, \rho \rangle$ is a poset with minimum ρ and the *forcing* relation $\Vdash$ is a binary relation on $P \times \mathcal{PV}$ such that $\alpha \Vdash p$ and $\alpha \leq \beta$ imply $\beta \Vdash p$ (monotonicity property). The forcing relation extends to arbitrary formulas of $\mathcal{L}$ as follows:

- $\alpha \Vdash \top$;
- $\alpha \nVdash \bot$;
- $\alpha \Vdash A \wedge B$ iff $\alpha \Vdash A$ and $\alpha \Vdash B$;
- $\alpha \Vdash A \vee B$ iff $\alpha \Vdash A$ or $\alpha \Vdash B$;
- $\alpha \Vdash A \rightarrow B$ iff, for every $\beta \in P$ such that $\alpha \leq \beta$, $\beta \Vdash A$ implies $\beta \Vdash B$;
- $\alpha \Vdash \neg A$ iff, for every $\beta \in P$ such that $\alpha \leq \beta$, $\beta \nVdash A$.

It is easy to prove that the monotonicity property holds for arbitrary formulas, i.e., $\alpha \Vdash A$ and $\alpha \leq \beta$ imply $\beta \Vdash A$. A formula A *is valid in a Kripke model* $\underline{K} = \langle P, \leq, \rho, \Vdash \rangle$ iff $\rho \Vdash A$. It is well-known that Intuitionistic Propositional Logic **Int** coincides with the set of formulas valid in all Kripke models [2].

A tableau calculus $\mathcal{T}$ works on *signed formulas*, namely formulas of $\mathcal{L}$ prefixed with one of the signs **T** or **F**. The semantics of formulas extends to signed formulas. Given a Kripke model $\underline{K} = \langle P, \leq, \rho, \Vdash \rangle$, $\alpha \in P$ and a signed formula H, α *realizes* H *in* $\underline{K}$ ($\underline{K}, \alpha \rhd H$) iff:

- $H \equiv \mathbf{T}A$ and $\alpha \Vdash A$;

- $H \equiv \mathbf{F}A$ and $\alpha \nVdash A$.

$\underline{K}$ realizes H ($\underline{K} \rhd H$) iff $\underline{K}, \alpha \rhd H$ for some $\alpha \in P$. H is *realizable* iff $\underline{K} \rhd H$, for some Kripke model $\underline{K}$. The above definitions extend in the obvious way to sets Δ of signed formulas; for instance, $\underline{K}, \alpha \rhd \Delta$ means that $\underline{K}, \alpha \rhd H$, for every $H \in \Delta$. By definition, $A \in \mathbf{Int}$ iff $\mathbf{F}A$ is not realizable.

We remark that, by the monotonicity property the $\mathbf{T}$-signed formulas are *persistent*, namely: $\underline{K}, \alpha \rhd \mathbf{T}A$ and $\alpha \leq \beta$ imply $\underline{K}, \beta \rhd \mathbf{T}A$. On the other hand, $\mathbf{F}$-signed formulas are not persistent.

In general, a tableau calculus $\mathcal{T}$ consists of a set of rules $\mathcal{R}$ of the form:

$$\frac{\Delta}{\Delta_1 \mid \cdots \mid \Delta_n}\, r$$

where Δ (the *premise* of r) and $\Delta_1, \ldots \Delta_n$ (the *consequences* of r) are sets of signed formulas of $\mathcal{L}$. A *proof table* for Δ is a tree τ such that:

- the root of τ is Δ;
- given a node Δ' of τ, the successors $\Delta_1, \ldots, \Delta_n$ of Δ in τ are the consequences of an instance of a rule of $\mathcal{R}$ having Δ' as premise.

A set Δ of signed formulas is *contradictory* if either $\mathbf{T}\bot \in \Delta$ or $\mathbf{F}\top \in \Delta$. When all the leaves of a proof table τ are contradictory, we say that τ is *closed*. A finite set of signed formulas Δ is *provable* in $\mathcal{T}$ iff there exists a closed proof table for Δ.

Let r be a rule with premise Δ and consequences $\Delta_1, \ldots, \Delta_n$. r is *sound* iff Δ realizable implies that there exists $k \in \{1, \ldots, n\}$ such that Δ_k is realizable. r is *invertible* iff r is sound and, for every $1 \leq k \leq n$, Δ_k realizable implies Δ realizable.

In this paper we refer to the calculus **Tab** of Figure 1, but one can consider *any* complete calculus for **Int**. In the formulation of the rules, we use the notation Δ, H as a shorthand for $\Delta \cup \{H\}$. Writing Δ, H in the premise of a rule we assume that $H \notin \Delta$. **Tab** is inspired to the calculus in [1] which uses the sign $\mathbf{F_c}$ besides the usual signs $\mathbf{T}$ and $\mathbf{F}$. In **Tab** the rules for $\mathbf{F_c}$ are translated by substituting $\mathbf{F_c}A$ with the equivalent signed formula $\mathbf{T}\neg A$. **Tab** turns out to be complete for **Int**, that is $A \in \mathbf{Int}$ iff $\{\mathbf{F}A\}$ is provable in **Tab**. More than this, the decision procedure discussed in [1] can be easily adapted to **Tab** preserving the time and space performances.

In this paper we provide invertible rules that can reduce the search space of the formula to be proved. This means that the decision procedure does not require to backtrack in the points where these rules are applied.

3 Replacement and simplification rules

First of all we recall the invertible rules introduced in [1]. Such rules allow us to simplify the signed formulas occurring in a node by replacing some of their

$$\frac{\Delta, \mathbf{T}(A \wedge B)}{\Delta, \mathbf{T}A, \mathbf{T}B}\mathbf{T}\wedge \qquad \frac{\Delta, \mathbf{F}(A \wedge B)}{\Delta, \mathbf{F}A \mid \Delta, \mathbf{F}B}\mathbf{F}\wedge \qquad \frac{\Delta, \mathbf{T}\neg(A \wedge B)}{\Delta_{\mathbf{T}}, \mathbf{T}\neg A \mid \Delta_{\mathbf{T}}, \mathbf{T}\neg B}\mathbf{T}\neg\wedge$$

$$\frac{\Delta, \mathbf{T}(A \vee B)}{\Delta, \mathbf{T}A \mid \Delta, \mathbf{T}B}\mathbf{T}\vee \qquad \frac{\Delta, \mathbf{F}(A \vee B)}{\Delta, \mathbf{F}A, \mathbf{F}B}\mathbf{F}\vee \qquad \frac{\Delta, \mathbf{T}\neg(A \vee B)}{\Delta, \mathbf{T}\neg A, \mathbf{T}\neg B}\mathbf{T}\neg\vee$$

$$\frac{\Delta, \mathbf{T}A, \mathbf{T}(A \to B)}{\Delta, \mathbf{T}A, \mathbf{T}B}\mathbf{T}{\to}Atom \quad \text{with } A \text{ an atom}$$

$$\frac{\Delta, \mathbf{F}(A \to B)}{\Delta_{\mathbf{T}}, \mathbf{T}A, \mathbf{F}B}\mathbf{F}{\to} \qquad \frac{\Delta, \mathbf{T}\neg(A \to B)}{\Delta_{\mathbf{T}}, \mathbf{T}A, \mathbf{T}\neg B}\mathbf{T}\neg{\to} \qquad \frac{\Delta_{\mathbf{T}}, \mathbf{T}(A \to B)}{\Delta_{\mathbf{T}}, \mathbf{T}\neg A \mid \Delta_{\mathbf{T}}, \mathbf{T}B}\mathbf{T}{\to}\text{-special}$$

$$\frac{\Delta, \mathbf{F}\neg A}{\Delta_{\mathbf{T}}, \mathbf{T}A}\mathbf{F}\neg \qquad \frac{\Delta, \mathbf{T}\neg\neg A}{\Delta_{\mathbf{T}}, \mathbf{T}A}\mathbf{T}\neg\neg$$

$$\frac{\Delta, \mathbf{T}((A \wedge B) \to C)}{\Delta, \mathbf{T}(A \to (B \to C))}\mathbf{T}{\to}\wedge \qquad \frac{\Delta, \mathbf{T}(\neg A \to B)}{\Delta_{\mathbf{T}}, \mathbf{T}A \mid \Delta, \mathbf{T}B}\mathbf{T}{\to}\neg$$

$$\frac{\Delta, \mathbf{T}((A \vee B) \to C)}{\Delta, \mathbf{T}(A \to p), \mathbf{T}(B \to p), \mathbf{T}(p \to C)}\mathbf{T}{\to}\vee \quad \text{with } p \text{ a new atom}$$

$$\frac{\Delta, \mathbf{T}((A \to B) \to C)}{\Delta_{\mathbf{T}}, \mathbf{T}A, \mathbf{F}p, \mathbf{T}(p \to C), \mathbf{T}(B \to p) \mid \Delta, \mathbf{T}C}\mathbf{T}{\to}{\to} \quad \text{with } p \text{ a new atom}$$

$$\frac{\Delta, \mathbf{T}A, \mathbf{F}A}{\Delta, \mathbf{T}\bot}contr_1 \qquad \frac{\Delta, \mathbf{T}A, \mathbf{T}\neg A}{\Delta, \mathbf{T}\bot}contr_2$$

where $\Delta_{\mathbf{T}} = \{\mathbf{T}A \mid \mathbf{T}A \in \Delta\}$

Fig. 1. The **Tab** calculus

subformulas either with $\bot$ or $\top$. Given a signed formula H and two formulas A and B, we denote with $H[B/A]$ the signed formula obtained by replacing every occurrence of A in H with B. If Δ is a set of signed formulas, $\Delta[B/A]$ is the set of signed formula $H[B/A]$ such that $H \in \Delta$.

It is easy to prove the following facts:

Lemma 1. *Let $\underline{K} = \langle P, \leq, \rho, \Vdash \rangle$ be a Kripke model, let H be a signed formula, A a formula and $\alpha \in P$.*

(i) If $\underline{K}, \alpha \rhd \mathbf{T}A$, then $\underline{K}, \alpha \rhd H$ iff $\underline{K}, \alpha \rhd H[\top/A]$.
(ii) If $\underline{K}, \alpha \rhd \mathbf{T}\neg A$, then $\underline{K}, \alpha \rhd H$ iff $\underline{K}, \alpha \rhd H[\bot/A]$.

Let us consider the following rules:

$$\frac{\Delta, \mathbf{T}A}{\Delta[\top/A], \mathbf{T}A}\text{Replace-}\mathbf{T} \qquad \frac{\Delta, \mathbf{T}\neg A}{\Delta[\bot/A], \mathbf{T}\neg A}\text{Replace-}\mathbf{T}\neg$$

By Lemma 1 it immediately follows that:

Theorem 1. *The rules* Replace-**T** *and* Replace-**T**¬ *are invertible.*

The above rules are the intuitionistic version of the analogous rules for classical tableaux discussed in [4]. After having applied a replacement rule, we can simplify the formulas in the consequence of the rule by means of the invertible rules in Figure 2.

$$
\begin{array}{llll}
\dfrac{\Delta}{\Delta[\bot/A \wedge \bot]}\,\mathsf{S}\wedge\bot &
\dfrac{\Delta}{\Delta[\bot/\bot \wedge A]}\,\mathsf{S}\bot\wedge &
\dfrac{\Delta}{\Delta[A/A \wedge \top]}\,\mathsf{S}\wedge\top &
\dfrac{\Delta}{\Delta[A/\top \wedge A]}\,\mathsf{S}\top\wedge \\
\dfrac{\Delta}{\Delta[A/A \vee \bot]}\,\mathsf{S}\vee\bot &
\dfrac{\Delta}{\Delta[A/\bot \vee A]}\,\mathsf{S}\bot\vee &
\dfrac{\Delta}{\Delta[\top/A \vee \top]}\,\mathsf{S}\vee\top &
\dfrac{\Delta}{\Delta[\top/\top \vee A]}\,\mathsf{S}\top\vee \\
\dfrac{\Delta}{\Delta[\top/\bot \to A]}\,\mathsf{S}\bot\to &
\dfrac{\Delta}{\Delta[\neg A/A \to \bot]}\,\mathsf{S}\to\bot &
\dfrac{\Delta}{\Delta[A/\top \to A]}\,\mathsf{S}\top\to &
\dfrac{\Delta}{\Delta[\top/A \to \top]}\,\mathsf{S}\to\top \\
\dfrac{\Delta}{\Delta[\bot/\neg\top]}\,\mathsf{S}\neg\top &
\dfrac{\Delta}{\Delta[\top/\neg\bot]}\,\mathsf{S}\neg\bot & &
\end{array}
$$

Fig. 2. Simplification rules

Now, we present the replacement rule for **F**-signed formulas [1]. We remark that, differently from classical logic, where the meaning of the signs **F** and **T** are opposite, in Intuitionistic Logic **F** and **T**-signed formulas have an asymmetric behavior. In particular, as noted in Section 2, **T**-signed formulas are persistent while **F**-signed formulas are not. Due to this asymmetry the replacement rule for **F**-signed formulas involves a notion of *partial substitution* which is weaker than the "full" substitution since the substitution does not act on propositional variables under the scope of implication or negation. Formally, given the formulas Z, A and B, we denote with $Z\{B/A\}$ the *partial substitution of A with B in Z* defined as follows:

- if $Z = A$, then $Z\{B/A\} = B$;
- if $Z = (X \odot Y)$, then $Z\{B/A\} = X\{B/A\} \odot Y\{B/A\}$, where $\odot \in \{\wedge, \vee\}$;
- if $Z = X \to Y$ or $Z = \neg X$ or Z is a propositional variable different from A, then Z{B/A}=Z.

We remark that differently from the "full" substitution rule denoted by square brackets, partial substitutions do not apply to subformulas with main connective $\to$ or $\neg$. For instance, while $((X \to Y) \vee Y)[\bot/Y]$ produces $(X \to \bot) \vee \bot$, the partial substitution $((X \to Y) \vee Y)\{Y/\bot\}$ yields $(X \to Y) \vee \bot$. Given a signed formula $\mathcal{S}Z$ with $\mathcal{S} \in \{\mathbf{T}, \mathbf{F}\}$, we denote with $\mathcal{S}Z\{B/A\}$ the signed formula $\mathcal{S}(Z\{B/A\})$. Given a set of signed formulas Δ, $\Delta\{B/A\}$ is the set containing $K\{B/A\}$ for every $K \in \Delta$.

It is easy to prove the following result [1]:

Lemma 2. *Let $\underline{K} = \langle P, \leq, \rho, \Vdash \rangle$ be a Kripke model, let $\alpha \in P$ and let H and $\mathbf{F}A$ be two signed formulas. If $\underline{K}, \alpha \rhd \mathbf{F}A$, then $\underline{K}, \alpha \rhd H$ iff $\underline{K}, \alpha \rhd H\{\bot/A\}$.*

Now, let us consider the rule:

$$\frac{\Delta, \mathbf{F}A}{\Delta\{\bot/A\}, \mathbf{F}A}\text{Replace-}\mathbf{F}$$

By the above Lemma 2 it immediately follows that:

Theorem 2. *The rule* Replace-**F** *is invertible.*

4 Propositional variables with constant sign

The replacement rules of Section 3 can be applied whenever a signed formula $\mathbf{T}A$, $\mathbf{T}\neg A$ or $\mathbf{F}A$ occurs in Δ. These rules together with the simplification rules in Figure 2 can considerably reduce the search space, as witnessed by the performances of PITP [1]. In this section, we exploit some conditions under which we can apply the replacement rules of Section 3 also to sets of signed formulas not explicitly containing $\mathbf{T}A$, $\mathbf{T}\neg A$ or $\mathbf{F}A$. The applicability of replacement rules can be foreseen evaluating the polarity of the propositional variables occurring in a signed formula.

Given a signed formula H and a propositional variable p, we introduce the notions $p \preceq^+ H$ (p *positively* occurs in H) and $p \preceq^- H$ (p *negatively* occurs in H). Hereafter we use $\mathcal{S}$ to denote either $\mathbf{T}$ or $\mathbf{F}$. The definition of $p \preceq^l H$, with $l \in \{+, -\}$ is by induction on the structure of H:

- $p \preceq^- \mathbf{F}p$ and $p \preceq^+ \mathbf{T}p$
- $p \preceq^l \mathcal{S}\top$ and $p \preceq^l \mathcal{S}\bot$
- $p \preceq^l \mathcal{S}q$, where q is any propositional variable such that $q \neq p$
- $p \preceq^l \mathcal{S}(A \odot B)$ iff $p \preceq^l \mathcal{S}A$ and $p \preceq^l \mathcal{S}B$, where $\odot \in \{\wedge, \vee\}$
- $p \preceq^l \mathbf{F}(A \to B)$ iff $p \preceq^l \mathbf{T}A$ and $p \preceq^l \mathbf{F}B$
- $p \preceq^l \mathbf{T}(A \to B)$ iff $p \preceq^l \mathbf{F}A$ and $p \preceq^l \mathbf{T}B$
- $p \preceq^l \mathbf{F}\neg A$ iff $p \preceq^l \mathbf{T}A$
- $p \preceq^l \mathbf{T}\neg A$ iff $p \preceq^l \mathbf{F}A$.

Given a set of signed formulas Δ, $p \preceq^l \Delta$ iff, for every $H \in \Delta$, $p \preceq^l H$.

Let us consider the following constructions over Kripke models. Given $\underline{K} = \langle P, \leq, \rho, \Vdash \rangle$ and a propositional variable p:

- $\underline{K}_p = \langle P, \leq, \rho, \Vdash' \rangle$, where $\Vdash' = \Vdash \cup \{(\alpha, p) \mid \alpha \in P\}$;
- $\underline{K}_{\neg p} = \langle P, \leq, \rho, \Vdash' \rangle$, where $\Vdash' = \Vdash \setminus \{(\alpha, p) \mid \alpha \in P\}$.

Note that, for every $\alpha \in P$, $\underline{K}_p, \alpha \rhd \mathbf{T}p$ and $\underline{K}_{\neg p}, \alpha \rhd \mathbf{T}\neg p$. It is easy to prove the following facts:

Lemma 3. *Let $\underline{K} = \langle P, \leq, \rho, \Vdash \rangle$ be a Kripke model, let H be a signed formula and let p be a propositional variable.*

(1) If $p\preceq^{+} H$ *then, for every* $\alpha \in P$, $\underline{K}, \alpha \rhd H$ *implies* $\underline{K}_p, \alpha \rhd H$.
(2) If $p\preceq^{-} H$ *then, for every* $\alpha \in P$, $\underline{K}, \alpha \rhd H$ *implies* $\underline{K}_{\neg p}, \alpha \rhd H$.

Proof. The proof easily goes by structural induction on H. As an example, we prove Point (1) for $H = \mathbf{T}(A \to B)$. Let us assume that $\underline{K}, \alpha \rhd \mathbf{T}(A \to B)$ and let β be any element of P such that $\alpha \leq \beta$ and $\underline{K}_p, \beta \rhd \mathbf{T}A$. To prove $\underline{K}_p, \alpha \rhd \mathbf{T}(A \to B)$ we have to show that $\underline{K}_p, \beta \rhd \mathbf{T}B$. Since $p\preceq^{+} \mathbf{F}A$ we have $\underline{K}, \beta \not\rhd \mathbf{F}A$, otherwise, by the induction hypothesis, $\underline{K}_p, \beta \rhd \mathbf{F}A$, in contradiction with the above assumption. Thus $\underline{K}, \beta \rhd \mathbf{T}A$ and, since $\underline{K}, \alpha \rhd \mathbf{T}(A \to B)$ and $\alpha \leq \beta$, it follows that $\underline{K}, \beta \rhd \mathbf{T}B$. Since $p\preceq^{+} \mathbf{T}B$, by the induction hypothesis $\underline{K}_p, \beta \rhd \mathbf{T}B$. $\square$

Now, let us consider the following rules:

$$\frac{\Delta}{\Delta[\top/p]}\ \mathbf{T}\text{-permanence} \qquad \text{where } p\preceq^{+} \Delta$$

$$\frac{\Delta}{\Delta[\bot/p]}\ \mathbf{T}\neg\text{-permanence} \qquad \text{where } p\preceq^{-} \Delta$$

Essentially these rules state that, if $p\preceq^{+} \Delta$ ($p\preceq^{-} \Delta$), we can consistently replace every occurrence of p in Δ with $\top$ ($\bot$, respectively). From the previous lemma it follows that:

Theorem 3. *The rules* **T**-permanence *and* **T**¬-permanence *are invertible.*

Proof. Let us consider the case of the rule **T**-permanence. We have to show that Δ is realizable iff $\Delta[\top/p]$ is realizable. Let us assume that Δ is realizable. Then, there exists a Kripke model $\underline{K} = \langle P, \leq, \rho, \Vdash \rangle$ and $\alpha \in P$ such that $\underline{K}, \alpha \rhd \Delta$. Since $p\preceq^{+} \Delta$, by Point (1) of Lemma 3, $\underline{K}_p, \alpha \rhd \Delta$ and, by definition of its forcing relation, $\underline{K}_p, \alpha \rhd \mathbf{T}p$. It follows that $\Delta, \mathbf{T}p$ is realizable and, by the soundness of the rule Replace-**T** (Lemma 1(i)), we get that $\Delta[\top/p]$ is realizable. Conversely, let us suppose that $\Delta[\top/p]$ is realizable and let $\underline{K} = \langle P, \leq, \rho, \Vdash \rangle$ be a Kripke model and $\alpha \in P$ such that $\underline{K}, \alpha \rhd \Delta[\top/p]$. Since p does not occur in $\Delta[\top/p]$, it holds that $p\preceq^{+} \Delta[\top/p]$. By Point (1) of Lemma 3, $\underline{K}_p, \alpha \rhd \Delta[\top/p]$. Since $\underline{K}_p, \alpha \rhd \mathbf{T}p$, by Lemma 1(i) $\underline{K}_p, \alpha \rhd \Delta$, hence Δ is realizable. The case of the rule **T**¬-permanence is similar. $\square$

We show an example of derivation where **T**-permanence works. Let

$$A = (\,(p \to q)\ \wedge\ ((\neg\neg r \to s) \to t)\ \wedge\ ((\neg\neg s \to t) \to p)\,)\ \to\ q$$

The formula A is classically valid but not intuitionistically valid[1]. To decide A, we have to search for a proof of $\mathbf{F}A$. Since $r\preceq^{+} \mathbf{F}A$, we can apply the rule **T**-permanence to get the set

$$\Delta_1\ =\ \{\,\mathbf{F}(\,(\,(p \to q)\ \wedge\ ((\neg\neg\top \to s) \to t)\ \wedge\ ((\neg\neg s \to t) \to p)\,)\ \to\ q\,)\,\}$$

[1] A is the formula SYJ211+1.001 of ILTP Library [5].

and, simplifying $\neg\neg\top \to s$ to s with the rules of Figure 2, we get:

$$\Delta_2 \ = \ \{\, \mathbf{F}(\,((p \to q) \ \wedge \ (s \to t) \ \wedge \ ((\neg\neg s \to t) \to p)\,) \ \to \ q\,)\,\}$$

Now, we can only proceed applying the rules $\mathbf{F}\to$ and $\mathbf{T}\wedge$ and we get:

$$\Delta_3 \ = \ \{\, \mathbf{T}(p \to q),\ \mathbf{T}(s \to t),\ \mathbf{T}((\neg\neg s \to t) \to p),\ \mathbf{F}q\,\}$$

The only rule applicable to Δ_3 is the branching rule $\mathbf{T}\to\to$ and we obtain the nodes

$$\begin{aligned} \Delta_4 \ &= \ \{\, \mathbf{T}(p \to q),\ \mathbf{T}(s \to t),\ \mathbf{T}\neg\neg s,\ \mathbf{F}a,\ \mathbf{T}(a \to p),\ \mathbf{T}(t \to a)\,\} \\ \Delta_5 \ &= \ \{\, \mathbf{T}(p \to q),\ \mathbf{T}(s \to t),\ \mathbf{T}p,\ \mathbf{F}q\,\} \end{aligned}$$

where a is a new propositional variable. Applying rules $\mathbf{T}\to \mathit{Atom}$ and contr_1 to Δ_5 we get a contradictory set. As for Δ_4, we have that $q \preceq^+ \Delta_4$, hence, applying $\mathbf{T}$-permanence we get

$$\Delta_6 \ = \ \{\, \mathbf{T}(p \to \top),\ \mathbf{T}(s \to t),\ \mathbf{T}\neg\neg s,\ \mathbf{F}a,\ \mathbf{T}(a \to p),\ \mathbf{T}(t \to a)\,\}$$

Simplifying we obtain

$$\Delta_7 \ = \ \{\, \mathbf{T}\top,\ \mathbf{T}(s \to t),\ \mathbf{T}\neg\neg s,\ \mathbf{F}a,\ \mathbf{T}(a \to p),\ \mathbf{T}(t \to a)\,\}$$

Now, $p \preceq^+ \Delta_7$, hence by $\mathbf{T}$-permanence and simplification, $\mathbf{T}(a \to p)$ reduces to $\mathbf{T}\top$ and we get

$$\Delta_8 \ = \ \{\, \mathbf{T}\top,\ \mathbf{T}(s \to t),\ \mathbf{T}\neg\neg s,\ \mathbf{F}a,\ \mathbf{T}(t \to a)\,\}$$

Now, we can we can only apply the $\mathbf{T}\neg\neg$ rule and we obtain the set

$$\Delta_9 \ = \ \{\, \mathbf{T}\top,\ \mathbf{T}(s \to t),\ \mathbf{T}s,\ \mathbf{T}(t \to a)\,\}$$

which is clearly not contradictory. Since in our derivation there is no backtrack point in the proof table, we conclude that $\mathbf{F}A$ is not provable.

If we disregard the rule $\mathbf{T}$-permanence, we have to begin the proof of $\mathbf{F}A$ by applying the rules $\mathbf{F}\to$ and $\mathbf{T}\wedge$ obtaining the set

$$\{\, \mathbf{T}(p \to q),\ \mathbf{T}((\neg\neg r \to s) \to t),\ \mathbf{T}((\neg\neg s \to t) \to p),\ \mathbf{F}q\,\}$$

At this point we have a backtracking point since the rule $\mathbf{T}\to\to$ can be applied to $\mathbf{T}((\neg\neg r \to s) \to t)$ or to $\mathbf{T}((\neg\neg s \to t) \to p)$.

We discuss in Section 6 the impact of the permanence rules on performances of PITP.

5 The rule F-permanence

In the previous section we have seen how the polarity of a propositional variable p can be used to predict if $\mathbf{T}p$ or $\mathbf{T}\neg p$ can be added to a deduction so to activate replacement and simplification rules. In this section we introduce a similar rule allowing us to predict when a **F**-signed propositional variable can be added to a deduction. Also in this case, such a prediction can be used to activate simplifications by applying the rule Replace-**F**. In this case the applicability of replacement rules can be foreseen evaluating if a propositional variable *weakly negatively occurs* in a set of signed formulas.

Given a propositional variable p and a signed formula H, let us define the relation $p \preceq^-_w H$ (p weakly negatively occurs in H) by induction on the structure of H:

- $p \preceq^-_w \mathcal{S}\top$ and $p \preceq^-_w \mathcal{S}\bot$
- $p \preceq^-_w \mathbf{F}A$ and $p \preceq^-_w \mathbf{T}\neg A$ for every A
- $p \preceq^-_w \mathbf{T}q$ if $q \neq p$
- $p \preceq^-_w \mathbf{T}(A \odot B)$ iff $p \preceq^-_w \mathbf{T}A$ and $p \preceq^-_w \mathbf{T}B$, where $\odot \in \{\wedge, \vee\}$
- $p \preceq^-_w \mathbf{T}(A \to B)$ iff $p \preceq^-_w \mathbf{T}B$.

We remark that $p \preceq^- H$ implies $p \preceq^-_w H$; on the other hand, the $\preceq^-_w$ relation permits weaker simplifications. Given a set Δ of signed formulas, we say that $p \preceq^-_w \Delta$ iff, for every $H \in \Delta$, $p \preceq^-_w H$.

Now let us consider the following construction over Kripke models. Given $\underline{K} = \langle P, \leq, \rho, \Vdash \rangle$ and a propositional variable p, let $\rho' \notin P$. By $\underline{K}^w_p$ we denote the Kripke model $\langle P', \leq', \rho', \Vdash' \rangle$ such that:

$$P' = P \cup \{ \rho' \} \qquad \leq' = \leq \cup \{ (\rho', \alpha) \mid \alpha \in P' \}$$
$$\Vdash' = \Vdash \cup \{ (\rho', q) \mid \rho \Vdash q \text{ and } q \neq p \}$$

Note that, for every signed formula H, $\underline{K}, \rho \rhd H$ iff $\underline{K}^w_p, \rho \rhd H$.

Lemma 4. *Let $\underline{K} = \langle P, \leq, \rho, \Vdash \rangle$ be a Kripke model, let H be a signed formula and let p be a propositional variable such that $p \preceq^-_w H$. Then, $\underline{K}, \rho \rhd H$ implies $\underline{K}^w_p, \rho' \rhd H$.*

Proof. Let us assume $\underline{K}, \rho \rhd H$. We prove $\underline{K}^w_p, \rho' \rhd H$ by induction on H. If $H = \mathbf{F}A$ then $\underline{K}^w_p, \rho \rhd \mathbf{F}A$, hence $\underline{K}^w_p, \rho' \rhd \mathbf{F}A$. If $H = \mathbf{T}q$ then $q \neq p$ (indeed $p \preceq^-_w \mathbf{T}p$ does not hold) and hence, by definition of $\Vdash'$, $\underline{K}^w_p, \rho' \rhd' H$. The cases $H = \mathbf{T}(A \wedge B)$ and $H = \mathbf{T}(A \vee B)$ easily follow by the induction hypothesis. Let $H = \mathbf{T}(A \to B)$; since $\underline{K}, \rho \rhd H$, we have $\underline{K}^w_p, \rho \rhd \mathbf{T}(A \to B)$. To prove that $\underline{K}^w_p, \rho' \rhd \mathbf{T}(A \to B)$, it only remains to show that $\underline{K}^w_p, \rho' \rhd \mathbf{T}A$ implies $\underline{K}^w_p, \rho' \rhd \mathbf{T}B$. If $\underline{K}^w_p, \rho' \rhd \mathbf{T}A$, then $\underline{K}^w_p, \rho \rhd \mathbf{T}A$, and this implies $\underline{K}, \rho \rhd \mathbf{T}A$. Since $\underline{K}, \rho \rhd \mathbf{T}(A \to B)$, we get $\underline{K}, \rho \rhd \mathbf{T}B$. Since $p \preceq^-_w \mathbf{T}B$, by induction hypothesis we conclude $\underline{K}^w_p, \rho' \rhd \mathbf{T}B$. The case $H = \mathbf{T}\neg A$ is similar. $\square$

Moreover, it is easy to prove:

Lemma 5. *Let H be a signed formula and p a propositional variable. If $p \preceq^-_w H$ then $p \preceq^-_w H\{\bot/p\}$.*

Proof. The proof is by induction on the structure of H. If $H = \mathbf{F}A$ or $H = \mathbf{T}\neg A$ or $H = \mathbf{T}q$ with q a propositional variable, the assertion immediately follows. If $H = \mathbf{T}(A \wedge B)$, then $p \preceq_w^- \mathbf{T}A$ and $p \preceq_w^- \mathbf{T}B$. By induction hypothesis, $p \preceq_w^- \mathbf{T}A\{\perp/p\}$ and $p \preceq_w^- \mathbf{T}B\{\perp/p\}$. Since $\mathbf{T}(A \wedge B)\{\perp/p\} = \mathbf{T}(A\{\perp/p\} \wedge B\{\perp/p\})$, it follows that $p \preceq_w^- H\{\perp/p\}$. The other cases are similar.

Now, let us consider the rule:

$$\frac{\Delta}{\Delta\{\perp/p\}}\ \mathbf{F}\text{-permanence} \qquad \text{where } p \preceq_w^- \Delta$$

Along the lines of the proof of Theorem 3, by lemmas 4 and 5 we get:

Theorem 4. *The rule* $\mathbf{F}$-permanence *is invertible.*

As an application of the above rule, let us consider the set

$$\Delta_1 \ = \ \{\, \mathbf{T}(p \vee q),\, \mathbf{F}(q \wedge r),\, \mathbf{F}(p \wedge r),\, \mathbf{F}(r \to q) \,\}$$

First of all, we notice that the propositional variables p, q and r do not occur in Δ_1 with constant sign, that is $x \not\preceq^+ \Delta_1$ and $x \not\preceq^- \Delta_1$ for every $x \in \{p, q, r\}$. Thus, the replacement rules discussed in the previous sections cannot be applied to Δ_1. On the other hand $r \preceq_w^- \Delta_1$, hence we can apply $\mathbf{F}$-permanence and we get the set

$$\Delta_2 \ = \ \{\, \mathbf{T}(p \vee q),\, \mathbf{F}(q \wedge \perp),\, \mathbf{F}(p \wedge \perp),\, \mathbf{F}(r \to q) \,\}$$

Applying the boolean simplifications to Δ_2 we get

$$\Delta_3 \ = \ \{\, \mathbf{T}(p \vee q),\, \mathbf{F}\perp,\, \mathbf{F}(r \to q) \,\}$$

Now, since $p \preceq^+ \Delta_3$, by applying the rules $\mathbf{T}$-permanence and the simplifications rules we obtain the set

$$\Delta_4 \ = \ \{\, \mathbf{T}\top,\, \mathbf{F}\perp,\, \mathbf{F}(r \to q) \,\}$$

which is non contradictory. Since the proof does not contain any branch we conclude that Δ_1 is not provable.

6 Timings

We devote this section to discuss the impact of the rules presented above. To this aim we have developed a Prolog implementation of the calculus of PITP [1] and we have tested how the performances are affected by the above rules. In particular, we compare the following theorem provers:

- BPPI (Basic Prolog Prover for Intuitionism) is the implementation of the calculus **Tab** extended with the rules Replace-$\mathbf{T}$, Replace-$\mathbf{T}\neg$ and the rules of Figure 2.

Prover	0-1s	1-10s	10-100s	100-600s	>600s
BPPI	1025(101.4)	51(158.9)	11(281.8)	4(972.3)	9(n.a.)
IPPI	856(165.4)	227(721.0)	12(416.2)	4(1745.5)	1(953.97)
EPPI	859(161.9)	226(710.7)	11(392.7)	4(1607.9)	0(0.0)
PITP	1085(11.3)	7(19.9)	3(165.3)	2(409.2)	3(20900.4)

Fig. 3. Timings

- IPPI (Intermediate Prolog Prover for Intuitionism) extends BPPI with the rules **T**-permanence and **T**¬-permanence.
- EPPI (Efficient Prolog Prover for Intuitionism) extends IPPI with the rule **F**-permanence.

Experiments[2] have been carried out along the lines of [5] and their results are summarized in Figure 3. The experiments have been performed on random generated formulas with 1024 connectives and a number of variables ranging from 1 to 1024. In every entry we indicate the number of formulas decided in the specified time range (expressed in seconds) and between brackets we put the total time required to decide them. The last row of the table refers to PITP [1] which is written in C++.

The results emphasize that for formulas decidable in few steps, the overhead of computing the variables with constant signs slows-down the prover, but when the formula to be decided requires a lot of computation, then the optimization is effective. As a matter of fact EPPI decides all the formulas within 10 minutes. The worth of our optimizations is confirmed when compared with the performances of PITP.

To conclude, in this paper we have presented a preliminary study about simplification rules in tableau calculi for Intuitionistic Logic. First of all, we remark that this topic has been scarcely studied in the literature, while it is central in classical theorem proving from its very beginning. Indeed, as far as we know, the rules presented in [4] are the only simplification rules for non classical logics described in the literature. These are the rules implemented in the calculus of PITP [1] and in our basic Prolog implementation BPPI. Now, considering the impact that the simplification rules have in implementation, we think that this topic deserves a deeper investigation.

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